Risk Pricing in Volatile Markets

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This study introduces a comprehensive pricing model addressing the evolving landscape of risk, incorporating market environment, correlation structures, and extreme jump risks. By integrating these elements into established option pricing models, the study offers a robust method for evaluating and managing emerging risks, particularly natural disasters and cyber threats. Application to real-world data, including hurricane losses and cyber risks, demonstrates its relevance and capability to derive market prices, allowing detailed comparisons of risk appetites across entities. Through calibration to market data, the model enables the definition of a unique probability distortion through a measure transformation, despite market incompleteness. This work enhances theoretical understanding of risk premiums and offers practical tools for risk management.

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1 Introduction

Over time, the risk landscape has changed significantly, leading to updates in the underlying models. Recently, the presence of extreme jump risks and the correlation of risks within a market have become more important. These factors are crucial in high volatility markets like oil and electricity, and in hedging against stock market crashes (e.g., Caldana and Fusai, 2013). Consequently, a significant part of the high equity and variance risk premiums observed today is compensation for these factors (Bollerslev and Todorov, 2011). Outside the classical financial market, these factors are also becoming more important, especially regarding societal risks. The increasing frequency and severity of natural disasters due to climate change, and cyber threats from global networking and digitalization, have broadened the scope of private risk-sharing beyond conventional property and casualty risks. This shift has moved away from relying solely on insurance companies, recognizing the collaborative nature of risk management. Insurance-linked securities now involve active participation from the capital markets. Additionally, the growing severity of losses, as seen with events like the Covid-19 pandemic, highlights the need for government involvement in managing such crises (e.g, Gründl et al., 2021; Braun et al., 2023).

The pricing of extreme risks is based on three key principles (Zanjani, 2002). First, shareholders may not be able to cover large, unexpected losses. Second, shareholders expect to be compensated for the risks they take, which ultimately affects policyholders. Third, policyholders care about the risk of the company going insolvent. Calibrating these factors to real-world data is challenging, and the resulting pricing often lacks a straightforward solution (e.g., Zanjani, 2002) or existing benchmark markets (e.g., Gründl et al., 2021).[1] However, most literature on insurable risks, including Zanjani (2002), still focuses on classic expected value theory. A model that consistently includes the market environment, correlation structures, and jump risks, accounting for higher-order factors, is still missing.

Based on this gap, the contribution of this paper is threefold. First, this study updates the four distinct risk categories from Cummins (2006) to reflect today's market conditions, assigning mathematical components to each. Second, the study introduces a new model for determining market risk premiums. This model, integrated into established option models, deviates from traditional approaches and is more suited to emerging risks. Grounded in the model proposed by Doherty and Garven (1986), it includes shareholders and policyholders and accounts for government frictions like taxes. It uses various option pricing literature to explain the characteristics of individual risk classes while maintaining a consistent framework. Central to this analysis are models from Margrabe (1978), focusing on the exchange dynamics of two risky assets, and Merton (1976), incorporating jump risks. Additionally, the closed-form solution by Cheang and Chiarella (2011) integrates these models effectively. The underlying actuarial principle is the measure transformation as in Gerber and Shiu (1994) and Wang (2000), which can be described as probability distortion. While this distortion is unique in simpler cases, it is not the case for extreme events. Third, the paper demonstrates

[1] The lack of high-frequency extreme risk data or models for calibrating frictions and risk premiums has led to the frequent use of unsuitable models in the literature, such as those by Fama and French (1993) or Fama and French (2015) for financial institutions (e.g., Cummins and Phillips, 2005). Therefore, it is necessary to merge or expand model approaches as discussed in more detail in Braun et al. (2023).

the practical application of this new methodology using real loss estimates. By calibrating model parameters with market data, the paper identifies market prices for various risks, focusing on jump components often transferred through cat bonds or cyber bonds. Despite friction and jump risk, precise market calibration allows for the calculation of unique probability distortion through measure transformation of individual markets or entities, which enables a comparison of the risk appetite across them.

This paper is structured as follows. Section 2 introduces the model, defines its boundaries, and describes the connection to existing pricing models. Section 3 shows the market application. Section 4 concludes.

2 Model

2.1 Risk categories

The risk categories are based on Cummins (2006) and can be summarized in four distinct classes:

- *Locally insurable:* Pertaining to independent risks characterized by moderate standard deviations per risk and a substantial number of policies, such as the U.S. market for personal automobile insurance. Local insurers can effectively cover these losses.

- *Globally insurable:* Encompassing risks that are locally dependent but globally independent, exemplified by the risk of tornadoes in the American Midwest versus Australia. Local insurers lack the capacity to cover such losses, but global reinsurers can. Consequently, these risks are diversifiable on a global scale through reinsurance.

- *Globally diversifiable:* Referring to risks with low frequency and very high severity, such as a 100 billion event in Florida or California. The capacity of insurance and reinsurance companies may prove insufficient to cover such events, but these risks can be globally diversified through participation in capital markets.

- *Globally undiversifiable:* Describing risks of such severity that they may resist global diversification, even through capital markets. For instance, a severe earthquake in Tokyo with losses ranging from 2.1 to 3.3 trillion. While global securities markets might absorb a fraction of such a loss, complete diversification of the full loss is unlikely.

Despite the economic coherence and comprehensiveness of these categories, which encompass all relevant private risk bearers, it is essential to address the underlying mathematical nuances. Cummins falls short in today's market environment, particularly for the last two categories.[2]

Local insurable risks follow a straightforward framework based on the law of large numbers, assuming the independence of losses within a loss portfolio. However, this independence

[2]Cummins defines the last two catastrophes as events that violate the principal insurability condition and may be globally diversifiable through capital markets if other conditions are satisfied. He does not specify any mathematical concepts, as he does for the first two categories.

does not apply when examining globally insurable risks from a local perspective. On a global scale, these risks exhibit no interdependence, allowing the creation of a loss portfolio of independent losses, which are then transferred to reinsurers due to their size. Thus, while there may be variations in the sizes of loss portfolios, local and global insurable risks are mathematically comparable and can be modeled using right-skewed and independent random variables (e.g., Eling, 2012).

Globally diversifiable and globally undiversifiable risks share characteristics of low frequency and high severity, with heavy tail events significantly influencing these risk profiles. They follow a structure of jump processes, such as a compounded Poisson process described by Merton (1976). A key distinction lies in the severity of globally undiversifiable risks, which can directly impact macroeconomic fundamentals. These risks uniquely correlate with the capital market and can trigger worldwide shocks, as seen with the Covid-19 pandemic (e.g., Gründl et al., 2021; Braun et al., 2023). Mathematically, in the first case, the occurrence of jumps and the jump size are uncorrelated, while in the latter case, there is a joint jump process with correlated jump sizes.

2.2 Option model

Inspired by Doherty and Garven (1986), a single-period model is considered. In $t = 0$ shareholders contribute equity S_0 and policyholders pay premiums P to cover the stochastic loss portfolio $\bar{L}$. The shareholder's opening cash flow is:

$$Y_0 = S_0 + P,$$

where the cash flow is invested at a risky rate $\bar{r}$. The terminal cash flow is:[3]

$$\bar{Y}_1 = (1 + \bar{r})\big(S_0 + P\big).$$

At the end of the period, the policyholders claim $\bar{L} \geq 0$, and the government (or other organizations such as supervisory authorities) claims frictional costs like monitoring, agency, tax, and liquidity $\bar{T}_1 \geq 0$. The policyholders receive the payment:

$$\begin{aligned}\bar{H}_1 &= min(\bar{L}, \bar{Y}_1) \\ &= \bar{Y}_1 - max(\bar{Y}_1 - \bar{L}, 0),\end{aligned}$$

and the additional frictional costs are:

$$\bar{T}_1 = max[\tau(\bar{Y}_1 - \bar{L}), 0],$$

where τ is the rate for frictional costs.[4]

[3]Doherty and Garven (1986) incorporate an adjustment to the premium investment by applying a coefficient for fundraising. This adjustment compensates for the temporal misalignment between the model period and the average delay between premium receipt and claims payment. For the sake of model simplicity, this adjustment is not included here.

[4]In the context of Doherty and Garven (1986), τ signifies the corporate tax rate, exclusively applied to income. Thus, $\bar{T}_1 = \max[\tau(\bar{Y}_1 - Y_0 + P - \bar{L}), 0]$. However, this depiction is excessively limiting, especially concerning capital market involvement and overall capital costs in the context of jump risk, as discussed in Zanjani (2002). Therefore, this aspect is further expounded upon here.

Both claims exhibit cash flows analogous to a European call option,[5] so the present values are:

$$H_0 = V(\bar{Y}_1) - C(\bar{Y}_1; \bar{L})$$
$$T_0 = \tau C(\bar{Y}_1; \bar{L}),$$

where $V(\cdot)$ is a present valuation operator and $C(A; B)$ is the current market value of a European call option with a terminal value A and exercise price B.

The present market value of the shareholder's return on equity, V_e, is the difference between the market value of the portfolio, $V(\bar{Y}_1)$, on the one side, and the present value of the policyholders' claims and the present value of the frictional costs on the other side:

$$\begin{aligned} V_e &= V(\bar{Y}_1) - H_0 - T_0 \\ &= C(\bar{Y}_1; \bar{L}) - \tau C(\bar{Y}_1; \bar{L}). \end{aligned}$$

In summary, shareholders hold a long position in a call option on the pre-frictional terminal value of the asset portfolio and a short position in a call option on the frictions of that portfolio.

Risk transfer prices are determined to yield a fair return to shareholders, achieved when the current market value of the equity claim equals the initial investment. As $\bar{Y}_1$ and Y_0 are functions contingent on P, the objective is to identify the premium P^* that satisfies:

$$\begin{aligned} V_e &= C(\bar{Y}_1(P^*); \bar{L}) - \tau C(\bar{Y}_1(P^*); \bar{L}) \\ &= S_0. \end{aligned}$$

Calculating P^* necessitates employing a suitable option-pricing framework. While Doherty and Garven (1986) establish pricing relationships within the discrete-time, risk-neutral-valuation framework of Rubinstein (1976), focusing on two special cases with (log-) normally distributed stochastic components[6], this study extends the analysis to stochastic processes and accounting for jump risks. Given the stochastic nature of the exercise price, conventional models like Black and Scholes (1973) are impractical. In a globally expanding world with heightened climate risks and population growth, emphasis on global diversification, including capital markets, and consideration of tail risk becomes increasingly crucial.

2.3 Pricing the option

Consider tradable assets X_1 and X_2 under a probability measure $\mathbb{P}$. Extending the option price formula from Black and Scholes (1973), Margrabe (1978) formulated a model allowing the exchange of two risky assets X_i. It is assumed that all returns come from capital gains

[5] The cash flow of a European call option is $CF_{call} = max(A - B, 0)$ with terminal value A and exercise price B.

[6] Distribution assumptions like the normal distribution prove inadequate, as highlighted by Eling (2012).

and that no dividends are distributed.[7] The dynamics for each asset are expressed as:

$$\frac{dX_i}{X_i} = \mu_i dt + \sigma_i dW_{i,t} \qquad i \in \{1, 2\},$$

where μ_i is the instantaneous expected return per unit time, σ_i is the instantaneous volatility per unit time and both assets follow a Brownian motion $dW_{i,t}$ with correlation ρ. This setting has the closed-form solution:

$$\begin{aligned} C(X_1, X_2) &= X_1\Phi(d_1) - X_2\Phi(d_2) \\ \text{with} \quad d_1 &= \frac{ln(\frac{X_1}{X_2}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

$T - t$ represents the difference between the exercise period and the present period, $\Phi(\cdot)$ is the cumulative standard normal density function, and $\sigma^2 = \sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2$.[8]

Globally diversifiable and globally undiversifiable risks are characterized by low frequency and high-severity events that fall beyond the scope of Margrabe (1978). The emergence of globally undiversifiable risk is inherently tied to economic fundamentals, indicating that not only does the loss portfolio but also the asset side exhibit a correlated downside risk. Modeling heavy tail risk involves incorporating jump processes, aligning with the conceptual framework established in Merton (1976). Unlike Margrabe, Merton's model does not consider the exchange of two risky assets but follows the methodology of Black and Scholes. Consequently, a synthesis of both approaches becomes essential in this context.

Let N_t be a Poisson process with a constant arrival rate of jumps λ, shared by both stocks. The bivariate process $\boldsymbol{Y} = (Y_1, Y_2)^T$ represents the jump sizes, taking values $\boldsymbol{y} = (y_1, y_2)^T \in \mathbb{R}^2$. The jump sizes Y_n are independently and identically distributed as multivariate normal $\mathcal{N}(\boldsymbol{\alpha}, \Sigma_{\boldsymbol{Y}})$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^T$, and the covariance matrix $\Sigma_{\boldsymbol{Y}}$ is given by:

$$\Sigma_{\boldsymbol{Y}} = \begin{pmatrix} \delta_1^2 & \rho_{\boldsymbol{Y}}\delta_1\delta_2 \\ \rho_{\boldsymbol{Y}}\delta_1\delta_2 & \delta_2^2 \end{pmatrix}$$

with $\rho_{\boldsymbol{Y}}$ representing the correlation between the jump sizes Y_1 and Y_2. The expected proportional common jump sizes are expressed as:

$$\kappa_i = \mathbb{E}_{\mathbb{P}}[\exp(Y_i) - 1)] = \int_{\mathbb{R}} [\exp(Y_i) - 1] m_{\mathbb{P}}(dy_i) \qquad i \in \{1, 2\},$$

where $m_{\mathbb{P}}(dy_i)$ is the density of Y_i (e.g., Merton, 1976).

Next, let $N_{i,t}$ be a Poisson process with a constant arrival rate of jumps λ_i and jump size Z_i, taking values $z_i \in \mathbb{R}$ for $i \in \{1, 2\}$. These processes are uncorrelated and specific to each

[7]The examination of dividend payout, as discussed in papers such as Cheang and Chiarella (2011), can be easily incorporated into the model. However, since it does not constitute a central core here, it is omitted to prevent additional complexity, but discussed in Appendix A.2.

[8]For $\sigma^2 = \sigma_1^2$ and $\sigma_2 = 0$, the formula from Black and Scholes (1973) is obtained.

asset. The idiosyncratic jump sizes are independently and identically normal-distributed as $\mathcal{N}(\alpha_{ii}, \delta_{ii}^2)$. The expected proportional unique jump sizes are given by:

$$\kappa_{Z_i} = \mathbb{E}_{\mathbb{P}}[\exp(Z_i) - 1] = \int_{\mathbb{R}} [\exp(Z_i) - 1] m_{\mathbb{P}}(dz_i) \qquad i \in \{1, 2\},$$

where $m_{\mathbb{P}}(dz_i)$ is the density of Z_i.

In summary, for each asset, the n-th common jumps $Y_{1,n}$ and $Y_{2,n}$ occur simultaneously, governed by the same Poisson arrival process N_t. These jointly occurring jumps can be linked to macroeconomic shocks in the system, representing globally undiversifiable risks. On the other hand, the m-th jump $Z_{1,m}$ or k-th jump $Z_{2,k}$, specific to the i-th asset, is influenced by the Poisson arrival process $N_{i,t}$. Jumps unique to each stock can be attributed solely to idiosyncratic shocks for that particular asset, defining globally diversifiable risks. The return dynamics of the assets can be expressed as:

$$\begin{aligned}\frac{dX_i}{X_i} =& (\mu_i - \lambda\kappa_i - \lambda_i \kappa_{Z_i})dt + \sigma_i dW_{i,t} \\ &+ \int_{\mathbb{R}} [\exp(y_i) - 1] p(dy_i, dt) + \int_{\mathbb{R}} [\exp(z_i) - 1] p(dz_i, dt) \qquad i \in \{1, 2\},\end{aligned}$$

where $p(\cdot, dt)$ is the Poisson measure. Poisson measures and the bivariate Wiener process are independent. The stock prices are given by the solution:

$$S_{i,t} = S_{i,0} \exp\left((\mu_i - \lambda\kappa_i - \lambda_i\kappa_{Z_i} - \frac{\sigma_i^2}{2})t + \sigma W_{i,t} + \sum_{n=1}^{N_t} Y_{i,n} + \sum_{m=1}^{N_{i,t}} Z_{i,m} \right) \qquad i \in \{1, 2\}.$$

To achieve a suitable and fair evaluation of the final payoff conditioned on information about the underlying asset prices, the probability measure $\mathbb{P}$ is transformed to $\mathbb{Q}$ using the transformation proposed by Esscher (1932), see Appendix A.1. After applying the transformation, the change in the intensity is defined by:

$$\begin{aligned}\tilde{\lambda} &= \lambda \exp(\upsilon) \mathbb{E}_{\mathbb{P}}[\exp(\boldsymbol{\gamma}^T \boldsymbol{Y})] \\ \tilde{\lambda}_1 &= \lambda_1 \exp(\upsilon_1) \mathbb{E}_{\mathbb{P}}[\exp(\beta_1 Z_1)] \\ \tilde{\lambda}_2 &= \lambda_2 \exp(\upsilon_2) \mathbb{E}_{\mathbb{P}}[\exp(\beta_2 Z_2)],\end{aligned}$$

and the expected jump sizes are transformed to:

$$\begin{aligned}\tilde{\kappa}_i &= \mathbb{E}_{\mathbb{Q}}[\exp(Y_i) - 1] \qquad i \in \{1, 2\} \\ \tilde{\kappa}_{Z_i} &= \mathbb{E}_{\mathbb{Q}}[\exp(Z_i) - 1] \qquad i \in \{1, 2\}.\end{aligned}$$

Hence, under $\mathbb{Q}$, the distribution of the jump sizes also changes. Y_n remains independently and identically multivariate normally distributed with $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + \Sigma_{\boldsymbol{Y}} \boldsymbol{\gamma}$; the jump sizes $Z_{i,k}$ are independently and identically normally distributed with $\tilde{\alpha}_{ii} = \alpha_{ii} + \delta_{ii}^2 \beta_i$ for $i \in \{1, 2\}$.

The parameters $\upsilon, \boldsymbol{\gamma}$ for the joint process, and υ_i, and β_i with $i \in \{1, 2\}$ for the distinct processes, are fundamental factors in the transition from $\mathbb{P}$ to $\mathbb{Q}$. The market, comprising stocks with jump components, is inherently incomplete following the sense of Harrison and Pliska (1981). When accounting for market prices of jump risks, multiple equivalent martingale measures emerge, leading to different option prices. For instance, if all factors equal zero, the scenario is akin to Merton (1976) where all jump risks are unpriced. If $\boldsymbol{\gamma} = \beta_i = 0$, while the other factors differ from zero, changes occur in jump-arrival intensities but not in jump-size distributions under the measure transformation. If υ and υ_i equal the logarithms of their moment-generating functions under $\mathbb{P}$, jump-arrival intensities remain unchanged despite changes in jump-size distributions under the measure transformation. Subsequently, attention is directed toward these parameters in the calibration to establish the market risk premium for the defined risk classes, underscoring their pivotal role in the model.

For the derivation of a closed-form option pricing formula considering these factors, the money account is assumed as the numeraire (see, e.g., Geman et al., 1995). A derivation of the option price formula can be found in Cheang and Chiarella (2011), so the proofs will not be repeated here. The notations were selected according to the paper. The dynamics of the asset prices under $\mathbb{Q}$ are expressed as:

$$\frac{X_i}{X_i} = rdt + \sigma_i d\tilde{W}i,t + \int_{\mathbb{R}} [\exp(y_i) - 1] q(dy, dt) + \int_{\mathbb{R}} [\exp(z_i) - 1] q(dz_i, dt) \qquad i \in \{1, 2\},$$

where $\tilde{W}_{i,t}$ denotes standard Brownian motion components under $\mathbb{Q}$, and q represents the Poisson measures under $\mathbb{Q}$. Therefore, the option price for the exchange of the two assets can be formulated as:

$$\begin{aligned} C(S_1, S_2) = \sum_k \sum_m \sum_n \exp\big(-(\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda})(T-t)\big) \frac{(\tilde{\lambda}_1(T-t))^k}{k!} \frac{(\tilde{\lambda}_2(T-t))^m}{m!} \frac{(\tilde{\lambda}(T-t))^n}{n!} \\ \times \Bigg[S_1 \exp\Big(-(\tilde{\lambda}_1 \tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)(T-t) + k\tilde{\alpha}_{11} + \frac{k\delta_{11}^2}{2} + n\tilde{\alpha}_1 + \frac{n\delta_1^2}{2}\Big) \Phi(d_{1,t,k,m,n}) \\ -S_2 \exp\Big(-(\tilde{\lambda}_2 \tilde{\kappa}_{Z_2} + \tilde{\lambda}\tilde{\kappa}_2)(T-t) + m\tilde{\alpha}_{22} + \frac{m\delta_{22}^2}{2} + n\tilde{\alpha}_2 + \frac{n\delta_2^2}{2}\Big) \Phi(d_{2,t,k,m,n}) \Bigg] \end{aligned}$$

where:

$$d_{1,t,k,m,n} = \frac{\ln(\frac{S_1}{S_2}) + (-\tilde{\lambda}(\tilde{\kappa}_1 - \tilde{\kappa}_2) - \tilde{\lambda}_1 \tilde{\kappa}_{Z_1} + \tilde{\lambda}_2 \tilde{\kappa}_{Z_2})(T-t) + \mu_{k,m,n} + \frac{\sigma_{k,m,n}^2 (T-t)}{2}}{\sigma_{k,m,n}\sqrt{T-t}}$$

$$d_{2,t,k,m,n} = d_{1,t,k,m,n} - \sigma_{k,m,n}\sqrt{T-t},$$

with:

$$\mu_{k,m,n} = k(\tilde{\alpha}_{1,1} + \frac{\delta_{1,1}^2}{2}) - m(\tilde{\alpha}_{2,2} + \frac{\delta_{2,2}^2}{2}) + n(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \frac{\delta^2}{2})$$

$$\sigma_{k,m,n}^2 = \sigma^2 + \frac{k\delta_{11}^2}{T-t} - \frac{m\delta_{22}^2}{T-t} + \frac{n\delta^2}{T-t},$$

where:

$$\delta^2 = \delta_1^2 + \delta_2^2 + \rho_Y \delta_1 \delta_2.$$

In the absence of jump risk, when $\tilde{\alpha}_i = \tilde{\alpha}_{ii} = 0$ and $\delta_i = \delta_{ii} = 0$, the jump intensity becomes zero, resulting in $\tilde{\kappa}_i = \tilde{\kappa}_{Z_i} = 0$ for $i \in \{1, 2\}$. Consequently, the option pricing formula of Margrabe (1978) is received, returning to the original model utilized at the beginning of the section.

Given that this paper examines a single-period model, the subsequent content adheres to the condition of $T - t = 1$.

2.4 Alternative models and limits

To facilitate a comparison with the newly proposed option model, a standard model from expected value theory and an extension by Zanjani (2002) is presented. To maintain simplicity, no discounting is applied in this section. The standard model acts as a benchmark in the empirical analysis of the paper and the premium is defined as the expected loss:

$$P = \mathbb{E}[\bar{L}].$$

This expression can be further expanded through linear scaling, for example, to accommodate frictional costs.

As the benchmark model does not incorporate insolvency or jump risk, t and consequently, does not include a risk premium, one can compute the risk premium by building the difference between the option model and the benchmark model:

$$risk\ premium = P_{\text{option model}} - P_{\text{Benchmark model}}.$$

To establish uniform coverage, the loss distribution for the benchmark model is truncated.

Zanjani (2002) introduce an extension to the standard model, focusing on the determination of (catastrophic) risk premiums. In his work, Zanjani highlights three primary considerations. Firstly, insurers may default due to heavy-tailed losses. Secondly, the cost associated with holding equity capital must be covered by premiums. Lastly, customers care about the insurer's risk of insolvency. According to Braun et al. (2023), the model can be summarized as follows:

$$P = \mathbb{E}[\bar{L}] - \mathbb{E}[D] + c.$$

Here, D denotes the difference between the expected payout and the realized payout in the event of insolvency:

$$D = max[\bar{L} - \bar{Y}_1, 0].$$

The cost of equity, denoted by c, is expressed as:

$$c = (\tau + r_{risk})S_0.$$

Here, r_{risk} signifies a risk premium determined by the correlation between loss and the capital market. This correlation is often assumed to be zero or close to zero (e.g., Cummins and Harrington, 1985; Froot et al., 1995; Zanjani, 2002). However, amidst the Covid-19 crisis, scholars like Gründl et al. (2021) and Braun et al. (2023) have begun to calibrate this term more generally. There is currently no definitive evidence regarding the specific form of this factor, e.g., whether it follows a linear pattern, e.g., Gründl et al. (2021) or exhibits concavity concerning the amount of risk, e.g., Braun et al. (2023).

Both alternative pricing models rely exclusively on first-order terms, while the new option model not only incorporates first-order terms but also defines second-order terms as significant price drivers. Notably, terms related to the market environment are absent in the alternative models. While both models overlook market and loss uncertainties, the benchmark model fails also to consider any insolvency risks, whereas the model by Zanjani (2002) concentrates solely on the insolvency risk for policyholders without addressing the corresponding risk for shareholders.[9]

All pricing models should produce identical outcomes when jump risks, insolvency risks, and other frictions are removed. In such scenarios, the premium should correspond to the expected loss. This market is distinguished by either an infinite amount of equity or no variance.

Lemma 1. *Under the assumptions of a frictionless market without insolvency and jump risk, it is necessary for all pricing models to satisfy:*

$$\lim_{S_0 \to \infty} P = \mathbb{E}[\bar{L}], \text{ and } \lim_{\sigma \to 0} P = \mathbb{E}[\bar{L}].$$

Proof. See Appendix A.3. □

With Lemma 1 established, the models exhibit convergence in a frictionless market without insolvency and jump risks. Additionally, the novel option model demonstrates its capability to incorporate insolvency risks and jump risks into pricing within market contexts where the alternative models fall short. The option model provides important insights into the potential risk transfer, especially concerning jump risks, by examining the limit behavior.

Lemma 2. *In a frictionless market without insolvency risk but with a positive probability of jump occurrences, the influence of jump risk is negligible, and consequently, it remains unpriced. The premium equals the expected loss:*

$$\lim_{S_0 \to \infty} P = \mathbb{E}[\bar{L}].$$

Proof. See Appendix A.3. □

Zanjani's model elucidates that in an insolvency-free market context, insurability of correlated jump risks is not feasible. In the presence of correlated jump risks, where $r_{risk} > 0$, it follows that as S_0 tends toward infinity, P also tends toward infinity. This underlines the limits of a linear (CAPM) approach when dealing with jump risks.

[9] To adequately account for these factors, they would need to be encompassed within r_{risk} to prevent their neglect. However, this is not included in the definition of the factor.

3 Risk premiums in the market

3.1 Locally insurable

The shareholders' return is contingent upon the performance of the S&P 500 index. According to Morningstar (2023), the annual total return over the past decade has averaged by approximately 9%, accompanied by a standard deviation of 15%. The assessment of locally insurable risk relies on US indemnity losses, as documented in Frees and Valdez (1998). The dataset comprises 1500 general liability claims, each representing the indemnity payment in USD. For scaling purposes, the data is divided by 1000, thus TUSD instead of USD is considered. This claims dataset is accessible through the R packages copula and evd. The expected loss per claim is 41.21 TUSD, with a standard deviation of 102.75 TUSD.

Figure 1 provides an overview of the premium to the expected loss for the benchmark model, the option model without frictions, and the option model involving frictions.[10] It is observed that the premium increases as the insolvency risk decreases and the higher the insolvency risk, the larger the risk premium. When shareholders supply sufficient equity to almost certainly cover the loss, the benchmark, and the option model converge to the expected loss (see Lemma 1). Frictions, particularly in tail risk, have an exponential effect on the premium. This is due to the substantial equity required for tail risk coverage, which consequently elevates costs. The onset of this growth varies depending on the magnitude of friction.

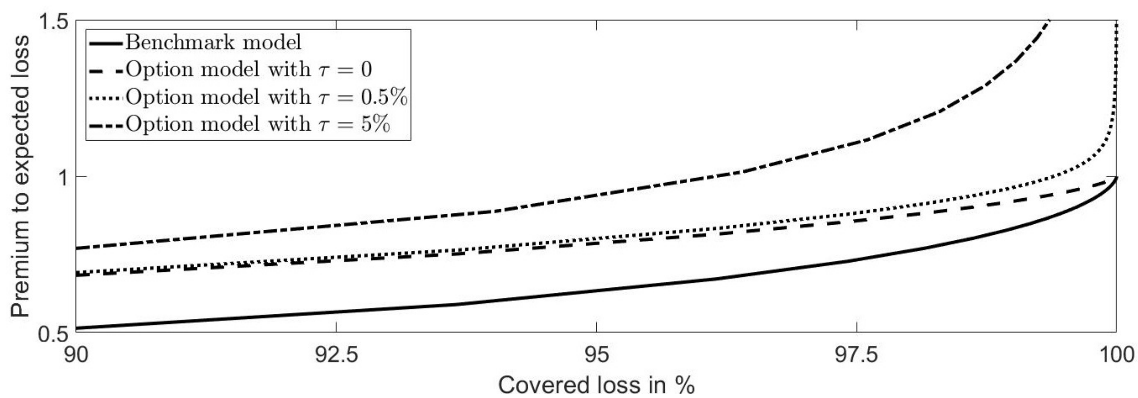
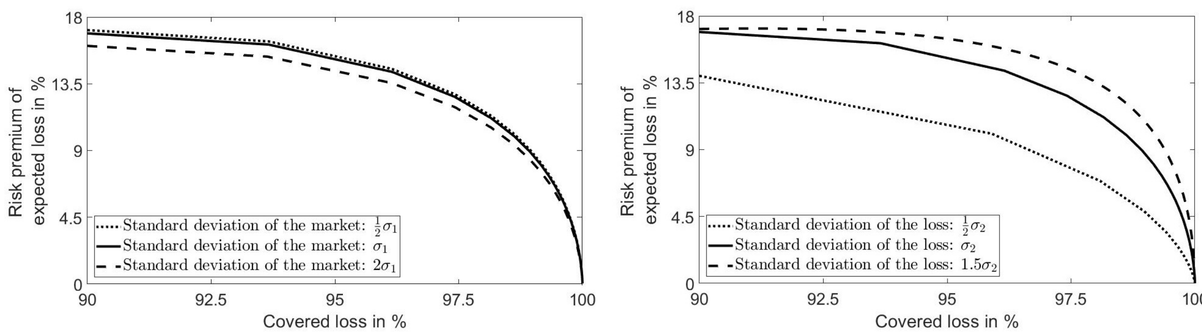


Figure 1: Premium for different market scenarios.

Figure 2 provides an overview of the risk premium in relation to the expected loss for (a) various market volatilities and (b) different loss volatilities. It is evident that as market volatility decreases, the risk premium increases. This is logically sound, as shareholders seek to invest their equity more securely, and policyholders, in turn, must pay a higher premium for the increased security. On the other hand, when the loss becomes more volatile, the risk premium also rises. This is reasonable, as higher loss volatility implies greater uncertainty and insolvency risk for shareholders, and with increasing uncertainty, shareholders demand a higher risk premium.

[10] A comparison with the model of Zanjani (2002) will also be implemented in a later version.



(a) Changing market volatility. (b) Changing loss volatility.

Figure 2: Risk premium for different market and loss volatilities.

Hence, the model presented reflects the price dynamics for locally insurable risks that are prevalent in the insurance markets. Firstly, the assessment of payment default risk is of importance for policyholders. Secondly, the pricing mechanisms in the insurance markets are influenced in particular by insolvency risk and frictional costs. Thirdly, the risk premium is subject to the dynamic interplay of market volatility, loss uncertainty, and insolvency risk.

3.2 Globally insurable

Data sourced from Grinsted et al. (2019) encompasses the majority of United States hurricanes dating back to the early 20th century. In this section, extreme events are omitted, excluding the 10% of the most potent hurricanes (see Braun et al., 2023). Given the vulnerability of Texas and North Carolina to hurricanes and the absence of historical data indicating a hurricane simultaneously impacting both states (uncorrelated risk), these two states are used for the analysis. Following the data, Texas exhibits an expected annual hurricane loss of USD 1,685 million, accompanied by a standard deviation 2.68 times the mean. North Carolina's expected annual hurricane loss amounts to USD 1,533 million, with a standard deviation of 3.34 times the mean. The combined portfolio of hurricane losses for both states anticipates an annual loss of USD 3,219 million, exhibiting a standard deviation of 2.05 times the mean.

Two scenarios are used for comparison: one where the respective portfolios are insured locally, and another where a reinsurer covers the combined portfolio, thereby diversifying the associated risks. Figure 3 shows the equity needed for the respective (re)insurer to underwrite the risk. The reinsurer exhibits the highest capital requirement but concurrently manages the largest policy volume. When aggregating the capital requirements of local insurers, the reinsurer consistently demonstrates lower capital requirements for the same portfolio, where this effect becomes stronger with more tail risk. This phenomenon is due to the global diversification of reinsurance and underlines a fundamental aspect of reinsurance, namely the improved efficiency of risk diversification through an increased capital base. This observation is corroborated by Figure 4. By minimizing the variance of the portfolio, the reinsurer can charge a lower risk premium, thereby conferring benefits to the end consumer. However, this advantage dissipates in the event of a hurricane affecting both Texas and North Carolina.

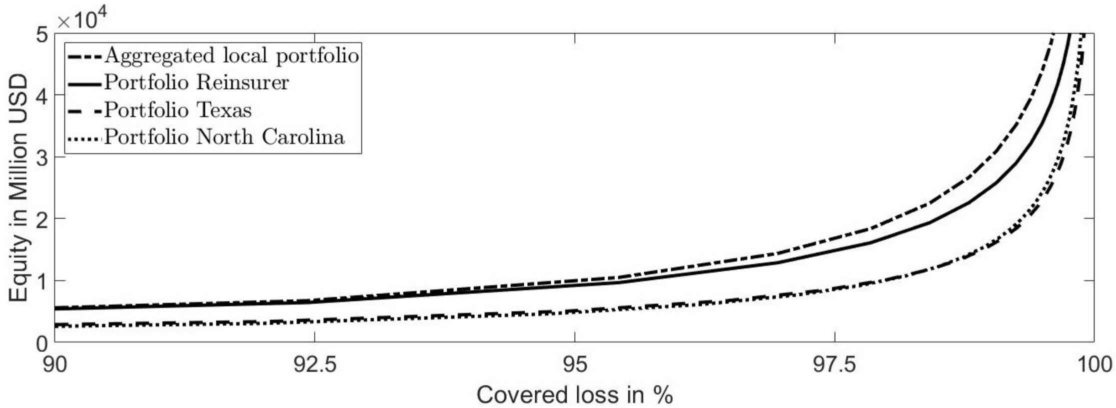


Figure 3: Equity for various local and global portfolios.

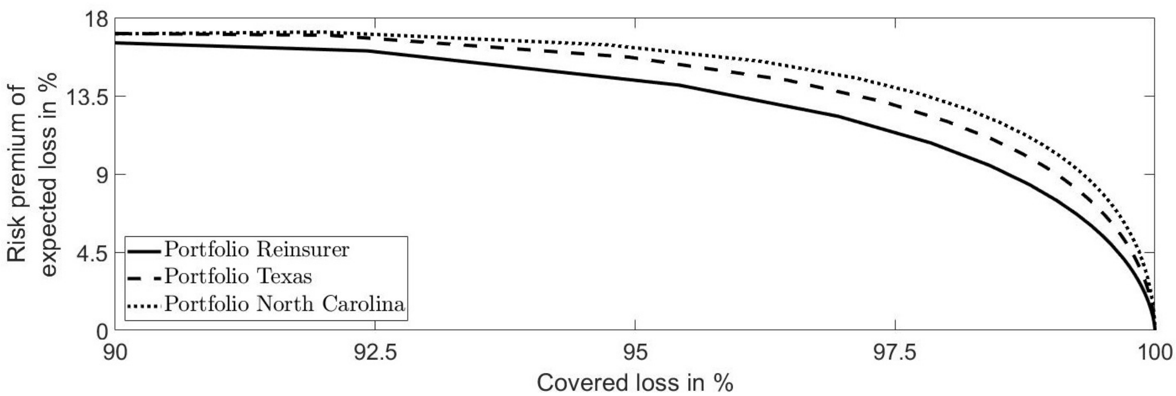


Figure 4: Risk premium for various local and global portfolios.

3.3 Globally diversifiable and globally undiversifiable

Globally diversifiable risk refers to the type of risk that is primarily covered by the capital market due to the amount of capital required which cannot be adequately provided by (re)insurers alone. This transfer of risk to the capital market typically involves the issuance of a cat bond. Shareholders of the bond pay a principal amount N to a trust account at time $t = 0$. In return, at time $t = 1$, they receive the risk-free rate earned from the trust account, a coupon payment C, and the principal, and need to pay the incurred losses (and any additional expenses). Therefore, the terminal cash flow for the shareholder is given by:

$$\bar{Y}_1 = (1 + r_f)N + C,$$

whereby $\bar{H}_1$ and $\bar{T}_1$ remain the same. To ensure that the initial cash flow for shareholders remains consistent throughout the model, the initial cash flow is:

$$Y_0 = N = S_0 + \frac{C}{1 + \omega}.$$

Here, ω represents a risk-adjusted discount rate, and $P = \frac{C}{1+\omega}$ denotes the present value premium paid by the policyholder (Braun et al., 2023).

Cat bonds typically feature an attachment point, beyond which they are activated. In this context, the trigger is set at the 90% quantile of annual hurricane losses, meaning that only losses exceeding USD 64,503 million are covered. Amounts below this threshold remain within the insurance market or are borne by policyholders. Consequently, the insurance market anticipates an expected annual hurricane loss of USD 15,966 million, with a standard deviation approximately 1.32 times the mean. It is assumed that the cat bond covers losses up to a maximum of USD 103,373 million, which corresponds approximately to the 95% quantile.

In the event of a hurricane triggering the cat bond, the loss, amounting to an expected loss of USD 26,949 million, falls into the capital market. This segment carries an expected annual hurricane loss with a standard deviation of around 0.53 times the expected value. The probability of such an event occurring is estimated at 10%. The hurricane data is retrieved from Grinsted et al. (2019).

The investment market is represented by the S&P 500, with the same assumptions as in the previous section. Based on historical market shocks from MFS (2023), it is assumed that a market crash occurs every 10 years with an average decline of 43.11% and a standard deviation of 0.34 times the expected value. Additionally, recent research found a frictional rate of 4.5% for cat bonds (Braun et al., 2023).

Three scenarios for the cat bond and the multiple of the bond, i.e., how many times the initial modeled expected loss investors receive in terms of the coupon, are examined, as shown in Table 1. If the multiple equals 1, it indicates a coupon without a risk premium. In Scenario 1, the probability of the jump in loss is positive, the capital market has no jump risk, and there are no frictions. In this case, the multiple is approximately 1.07, indicating a market risk premium of 7% of the expected loss. If the capital market also has a risk of a crash, the market risk premium falls to around 2%. This is consistent with previous results showing that the more volatile the market, the lower the risk premium. In the last scenario, frictional costs are added, and the market risk premium rises to 58%. This aligns with previous research indicating that frictional costs account for a large part of the premium for extreme risks (e.g., Braun et al., 2023).

According to Artemis (2024), the average multiple for the cat bond market for Q2 2024 is 4, meaning the risk premium is 300% of the expected loss. However, the market risk premium calculated here is only around 172%. Assuming that the underlying cat bond is priced consistently with the market, a probability distortion can be assumed. The parameters of the measure transformation must be calibrated accordingly. Scenarios 4 and 5 show the results of the marginal solutions, where the distortion occurs only in the probability of the jump (Scenario 4) or in the jump size (Scenario 5). In theory, the distortion can also occur in both, but the parameters are linear to each other and add up to the same value and are therefore unique, as shown in Figure 5. Consequently, the user can decide where the distortion is larger, e.g., depending on the modeling risk.

Based on the calibrations, a Covid-19 bond is priced, demonstrating that this risk class cannot be transferred to the capital market without prior adjustments, such as government support (e.g., Braun et al., 2023). The underlying assumption is a pandemic occurring twice in a century, as exemplified by the Spanish flu and Covid-19. The S&P 500 serves as the underlying market, with the jump size derived from the previous chapter. Business interruptions, which are part of property insurance contracts (APICA, 2020), are considered as the damage caused by the pandemic. Therefore, the losses from the previous chapter are used as the base loss. To ensure comparability with the cat bond, the same jump size is assumed. This entails adopting a lower limit for the jump process, acknowledging that in the real world, undiversifiable jumps are significantly higher (e.g., APICA, 2020). Essentially, the jump probabilities are reduced, but the jump itself is included as a joint process with correlated jump sizes.

Table 1 shows the multiple for the fictitious Covid-19 bond. Scenario 1 is not available due to the joint process. In Scenario 2, the multiple more than triples, although the probability of occurrence is only one-fifth. If frictional costs are included, the multiple doubles. Although it can be assumed that the modeling risk for pandemics is greater than for natural disasters due to the limited data and is therefore more of a lower limit, it is now assumed that the probability distortion is identical in the risk classes. Scenarios 4 and 5 show that the resulting market risk premium is more than 14 times the expected loss. For comparison, the largest historical market multiple was 7.5 at the beginning of the cat bonds in 2001 (Artemis, 2024). Accordingly, it can be concluded that given the underlying market, a transfer of pandemic risks through the capital market is not feasible (e.g., Gründl et al., 2021). Possible interventions could include reducing frictional costs; if Scenario 4 and 5 are assumed with no frictional costs, the multiple reduces to around 6. Even though a market risk premium of 500% of the expected damage is very high, it represents a significant reduction. However, it must be noted that a lower limit for the jump size was used here. If the jump size increases, the relative multiple may remain stable, but the absolute values become unaffordable. Additionally, the correlation of the jump sizes was omitted here. Since only extreme events with a comparably low variance that occur together due to the Poisson process are considered, correlation has little influence on the multiple. This may differ in absolute terms, as will be discussed in the next chapter.

In summary, this chapter has demonstrated that the new pricing model accurately reflects the key relationships between risk and the market. It can calculate market risk premiums based on the market environment, accommodating both hard and soft market conditions. Additionally, it can price jump risks, whether independent or joint. With appropriate calibration, the risk appetite of the market can be determined, as discussed in the next section. Beyond the five scenarios considered, many other combinations are conceivable. For instance, integrating market risk with jump risks into locally and globally insurable risks could further enhance the model's realism.

Scenario		cat bond multiple	Covid-19 Bond multiple
1	$\lambda_2 = 0.1$	1.0689	-
2	$\lambda_1 = \lambda_2 = 0.1$ or $\lambda = 0.02$	1.0201	3.8122
3	$\lambda_1 = \lambda_2 = 0.1$ or $\lambda = 0.02$ $\tau = 0.045$	1.5783	6.6439
4	$\lambda_1 = \lambda_2 = 0.1$ or $\lambda = 0.02$ $\tau = 0.045$ $\upsilon_2 = \upsilon = -0.7078$	4	15.1930
5	$\lambda_1 = \lambda_2 = 0.1$ or $\lambda = 0.02$ $\tau = 0.045$ $\beta_2 = \gamma_2 = -0.4019$	4	17.2877
6	$\lambda_1 = \lambda_2 = 0.1$ or $\lambda = 0.02$ $\upsilon_2 = \upsilon = -0.7078$	2.8752	9.4690
7	$\lambda_1 = \lambda_2 = 0.1$ or $\lambda = 0.02$ $\beta_2 = \gamma_2 = -0.4019$	2.8775	10.8599

Table 1: Multiples for the cat bond and the Covid-19 bond without and with measurement change.

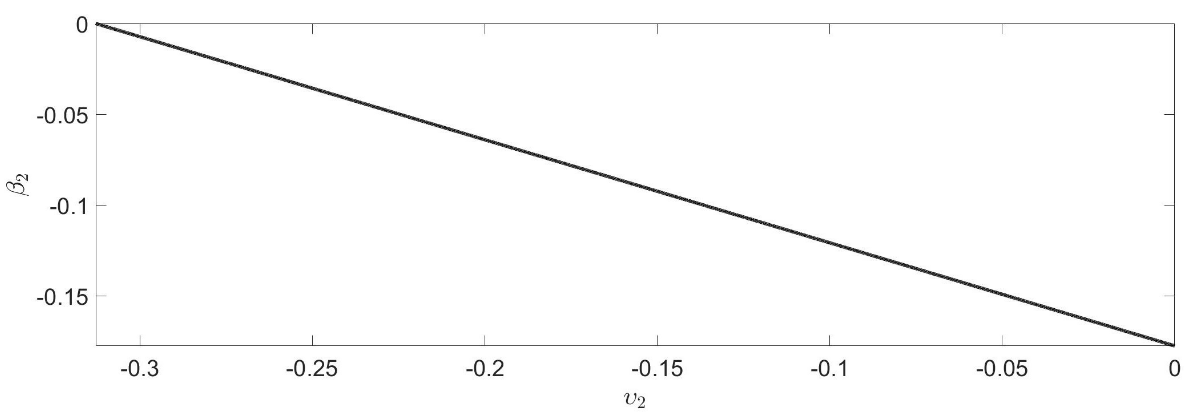


Figure 5: Linear relationship of υ_2 and β_2 for a target multiple of 4 for the cat bond.

3.4 Outlook - Risk appetite

The next step extends the analysis to comprehend the risk appetite of specific market entities for risks in the third category. The calibration again utilizes the hurricane data from Grinsted et al. (2019), focusing specifically on the previously excluded 10% severe cases. Additionally, a recently acquired private dataset from RMS and GallagherRe, representing the global cyber risk market, will be used.

The aim is to understand the stability of the measure transformation with respect to individual participants. While the previous chapter demonstrated how a probability distortion can be calibrated based on the market, this section will examine the sensitivity of the measurement in the extreme risk market by considering both NatCat and cyber risks. Two market representatives will be analyzed: (1) SwissRe, which issued a cat bond and a cyber bond one year apart, and (2) Beazley, which issued two Cyber Bonds with different characteristics within a short period.

By cross-comparing these four bonds, it is possible to understand how individual market participants differentiate their risk appetite over time and between individual risks. Since the processes differ from each other, such as in expected returns, attachment points, etc., the probability distortion described by the measure transformation serves as a clear measure to understand this risk appetite. This demonstrates that the model can be used not only for pricing but also for comparing market participants and risks.

4 Conclusion

This study addresses the evolving landscape of risk management by introducing a comprehensive pricing model that accounts for market environment, correlation structures, and extreme jump risks. The updated risk categories and new model for determining market risk premiums fill a significant gap in the literature, providing a more accurate framework for contemporary risk assessment. By integrating these elements into established option pricing models, the study offers a robust method for evaluating and managing emerging risks, particularly those associated with natural disasters or cyber threats.

The application of this model to real-world data, including hurricane losses and cyber risks, demonstrates its practical relevance and capability to derive market prices for various risks. Through calibration to market data, this model enables the definition of a unique probability distortion through a measure transformation, despite market incompleteness. This approach allows for a detailed comparison of risk appetites across different market entities, highlighting the importance of precise market calibration even in the presence of friction and jump risks.

The results underscore the necessity of incorporating higher-order factors into risk pricing models to reflect the complexities of modern markets. This work not only enhances the theoretical understanding of risk premiums but also provides practical tools for risk management in both financial and non-financial sectors.

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A Appendix

A.1 Measure changes using the Esscher transformation

Define an asset as:

$$S_t = S_0 \exp(X_t),$$

where $X_t t \geq 0$ is a stochastic process characterized by stationary and independent increments, and $X_0 = 0$. Furthermore, let:

$$F_{X_t}(x) = \mathbb{P}(X_t \leq x)$$

be the cumulative distribution function, and:

$$M_{\mathbb{P},X_t}(u) = \mathbb{E}[\exp(uX_t)]$$

represent the moment-generating function of the random variable X_t under the measure $\mathbb{P}$. Thus:

$$M_{\mathbb{P},X_t}(u) = \int_{-\infty}^{\infty} \exp(ux) f(x,t) dx,$$

where $f(x,t)$ is the continuous density of X_t.[11] Building upon the transformation proposed by Esscher (1932), a transformed density for X_t is:

$$\begin{aligned} f(x,t,h) &= \frac{\exp(hx) f(x,t)}{\int_{-\infty}^{\infty} \exp(hy) f(y,t) dy} \\ &= \frac{\exp(hx) f(x,t)}{M_{\mathbb{P},X_t}(h)} \end{aligned}$$

where h is the transformation parameter. The corresponding moment-generating function is given by:

$$\begin{aligned} M_{Q,X_t}(u) &= \int_{-\infty}^{\infty} \exp(ux) f(x,t,h) dx \\ &= \frac{M_{\mathbb{P},X_t}(u+h)}{M_{\mathbb{P},X_t}(u)}. \end{aligned}$$

Subsequently, the Esscher transformation is derived for the three significant processes in this study. The analytical findings align with prior literature, exemplified by works such as Gerber and Shiu (1994) and Runggaldier (2003), where, for instance, Runggaldier transforms these measures utilizing the Radon-Nikodym theorem. In the provided examples, the time component is disregarded, as it is not needed in this context.

[11] For a discrete distribution, the integral can be replaced by a sum.

Normal distribution: Assuming $X_t = Y_t$, where Y_t is a normally distributed random variable with a mean of μ and a variance of σ^2. The moment-generating function is expressed as:

$$M_{\mathbb{P},X_t}(u) = \exp(u\mu + \frac{1}{2}\sigma^2 u^2).$$

Through the Esscher transformation, the resulting expression for the moment-generating function under the new measure $\mathbb{Q}$ is:

$$M_{\mathbb{Q},X_t}(u) = \exp\Big(u(\mu + h\sigma^2) + \frac{1}{2}\sigma^2 u^2\Big).$$

Consequently, the new mean under $\mathbb{Q}$ can be defined as $\tilde{\mu} = \mu + h\sigma^2$. The transformed normal distribution under $\mathbb{Q}$ remains a normal distribution with mean $\tilde{\mu}$ variance σ^2.

Proof.

$$\begin{aligned}
\frac{M_{\mathbb{P},X_t}(u+h)}{M_{\mathbb{P},X_t}(u)} &= \frac{\exp\big((u+h)\mu + \frac{1}{2}\sigma^2(u+h)^2\big)}{\exp(u\mu + \frac{1}{2}\sigma^2 u^2)} \\
&= \exp\Big((u+h)\mu + \frac{1}{2}\sigma^2(u+h)^2 - (u\mu + \frac{1}{2}\sigma^2 u^2)\Big) \\
&= \exp\Big((h\mu + \frac{1}{2}\sigma^2 h^2 + \sigma^2 uh\Big) \\
&= \exp\Big((h(\mu + \sigma^2 u) + \frac{1}{2}\sigma^2 h^2\Big)
\end{aligned}$$

□

Poisson distribution: Assume $X_t = kN_t$, where N_t is a Poisson process with intensity λ, and k is a constant. The moment-generating function is defined as:

$$M_{\mathbb{P},X_t}(u) = \exp\Big(\lambda(\exp(ku) - 1)\Big)$$

Through the Esscher transformation, the resulting expression for the moment-generating function under the new measure $\mathbb{Q}$ is:

$$M_{\mathbb{Q},X_t}(u) = \exp\Big(\lambda\exp(hk)(\exp(ku) - 1)\Big)$$

Consequently, the intensity under $\mathbb{Q}$ can be defined as $\tilde{\lambda} = \lambda\exp(hk)$. The transformed Poisson process under $\mathbb{Q}$ remains a Poisson process with intensity $\tilde{\lambda}$.

Proof.

$$\begin{aligned}
\frac{M_{\mathbb{P},X_t}(u+h)}{M_{\mathbb{P},X_t}(u)} &= \frac{\exp\big(\lambda(\exp(k(u+h)) - 1)\big)}{\exp\big(\lambda(\exp(ku) - 1)\big)} \\
&= \exp\Big(\lambda(\exp(k(u+h)) - 1) - \lambda(\exp(ku) - 1)\Big) \\
&= \exp\Big(\lambda(\exp(ku)\exp(kh)) - \lambda\exp(ku)\Big) \\
&= \exp\Big(\lambda\exp(ku)(\exp(kh) - 1)\Big)
\end{aligned}$$

$\square$

Compounded Poisson process: Assume a compounded Poisson process $X_t = \sum_{i=1}^{N_t} Y_t$, where N_t is a Poisson process with intensity λ, and Y_t represents a normally distributed jump size with mean μ and variance σ^2. The moment-generating function is defined as:

$$\begin{aligned}
M_{\mathbb{P},X_t}(u) &= \mathbb{E}[\exp(u\sum_{i=1}^{N_t} Y_i)] \\
&= \exp\big(\lambda(M_{\mathbb{P},Y_t}(u) - 1)\big)
\end{aligned}$$

Through the Esscher transformation, the resulting expression for the moment-generating function under the new measure $\mathbb{Q}$ is:

$$M_{\mathbb{Q},X_t}(u) = \exp\Big(\lambda \exp(v) M_{\mathbb{P},Y_t}(h)(M_{\mathbb{Q},Y_t}(u) - 1)\Big)$$

Consequently, the intensity under $\mathbb{Q}$ can be defined as $\tilde{\lambda} = \lambda \exp(v) M_{\mathbb{P},Y_t}(h)$, and the new mean of the jump size under $\mathbb{Q}$ can be defined as $\tilde{\mu} = \mu + h\sigma^2$. The transformed compounded Poisson process under $\mathbb{Q}$ remains a compounded Poisson process with intensity $\tilde{\lambda}$ and mean jump size $\tilde{\mu}$ and variance σ^2.

Proof.

$$\begin{aligned}
\frac{M_{\mathbb{P},X_t}(u+h)}{M_{\mathbb{P},X_t}(u)} &= \frac{\exp\big(\lambda(M_{\mathbb{P},Y_t}(u+h) - 1)\big)}{\exp\big(\lambda(M_{\mathbb{P},Y_t}(u) - 1)\big)} \\
&= \exp\Big(\lambda(M_{\mathbb{P},Y_t}(u+h) - 1) - \lambda(M_{\mathbb{P},Y_t}(u) - 1)\Big)
\end{aligned}$$

Given the moment-generating function of a normally distributed random variable, one obtains:

$$\begin{aligned}
M_{\mathbb{P},Y_t}(u+h) &= \exp\Big((u+h)\mu + \frac{1}{2}\sigma^2(u+h)^2\Big) \\
&= \exp\Big(u\mu + h\mu + \frac{1}{2}\sigma^2 u^2 + \frac{1}{2}\sigma^2 h^2 + \sigma^2 uh\Big) \\
&= \exp\Big(u\mu + \frac{1}{2}\sigma^2 u^2 + h(\mu + \sigma^2 u) + \frac{1}{2}\sigma^2 h^2\Big) \\
&= M_{\mathbb{P},Y_t}(u) M_{\mathbb{Q},Y_t}(h)
\end{aligned}$$

Therefore:

$$\begin{aligned}
\frac{M_{\mathbb{P},X_t}(u+h)}{M_{\mathbb{P},X_t}(u)} &= \exp\Big(\lambda(M_{\mathbb{P},Y_t}(u) M_{\mathbb{Q},Y_t}(h) - 1) - \lambda(M_{\mathbb{P},Y_t}(u) - 1)\Big) \\
&= \exp\Big(\lambda M_{\mathbb{P},Y_t}(u) M_{\mathbb{Q},Y_t}(h) - \lambda M_{\mathbb{P},Y_t}(u)\Big) \\
&= \exp\Big(\lambda M_{\mathbb{P},Y_t}(u)(M_{\mathbb{Q},Y_t}(h) - 1)\Big)
\end{aligned}$$

To enable the variation of only the intensity while keeping the jump size constant, a parameter $\exp(\upsilon)$ is introduced. This parameter aligns with the Esscher transformation. and can be viewed as a generalization. It is used to allow the poission process to be transformed without changing the jumpsizes, for example, the modeling risk lies in the frequency rather than the severity of loss. $\square$

A.2 Influence of dividends on premiums

Following Cheang and Chiarella (2011), both assets may yield a dividend return denoted as $\xi_i, i \in \{1, 2\}$. In the context of this study, wherein S_2 represents the loss, dividend payments do not apply to this asset, resulting in $\xi_1 \geq 0$ and $\xi_2 = 0$. Consequently, the formulation of the option price for the exchange of the two assets, accounting for dividends, can be formulated as:

$$
\begin{aligned}
C(S_1, S_2) &= \sum_k \sum_m \sum_n \exp\big(- (\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda})\big) \frac{(\tilde{\lambda}_1)^k}{k!} \frac{(\tilde{\lambda}_2)^m}{m!} \frac{(\tilde{\lambda})^n}{n!} \\
\times \Bigg[& S_1 \exp\Big(- (\xi_1 + \tilde{\lambda}_1 \tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1) + k\tilde{\alpha}_{11} + \frac{k\delta_{11}^2}{2} + n\tilde{\alpha}_1 + \frac{n\delta_1^2}{2} \Big) \Phi(d_{1,t,k,m,n}) \\
& - S_2 \exp\Big(- (\tilde{\lambda}_2 \tilde{\kappa}_{Z_2} + \tilde{\lambda}\tilde{\kappa}_2) + m\tilde{\alpha}_{22} + \frac{m\delta_{22}^2}{2} + n\tilde{\alpha}_2 + \frac{n\delta_2^2}{2} \Big) \Phi(d_{2,t,k,m,n}) \Bigg]
\end{aligned}
$$

where:

$$
d_{1,t,k,m,n} = \frac{\ln(\frac{S_1}{S_2}) + (-\tilde{\lambda}(\tilde{\kappa}_1 - \tilde{\kappa}_2) - \tilde{\lambda}_1 \tilde{\kappa}_{Z_1} + \tilde{\lambda}_2 \tilde{\kappa}_{Z_2} - \xi_1) + \mu_{k,m,n} + \frac{\sigma_{k,m,n}^2}{2}}{\sigma_{k,m,n}\sqrt{T-t}}.
$$

The other terms remain unchanged.

Given the indemnity losses in the US from the example in Section 3.1. The dividend yield of the S&P 500 index was at the end of 2022 by 1.78%, whereas historical dividend yields for the S&P 500 index have typically ranged from between 3% to 5% (Ross, 2023). Figure 6 illustrates the premium differences between dividends and no dividends. The pattern resembles that seen with frictional costs. This suggests that dividends in alternative investments are a price determinant for insurance contract premiums.

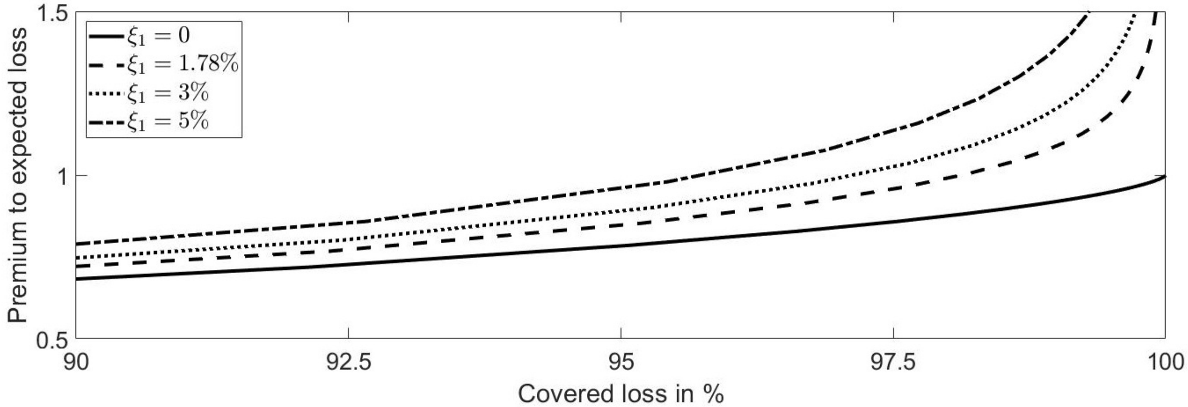

Figure 6: Premium for different dividends.

A.3 Proofs

Lemma 1

Proof. In the benchmark model, the proof is straightforward. In the extension, given the absence of insolvency risk, $E[D] = 0$. Moreover, without friction and jump risk, $c = 0$. Hence, $P = \mathbb{E}[\bar{L}]$. In the option model, when jump risk is absent, the following relationships hold:

$$\begin{aligned} & & Y_0\Phi(d_1) - \mathbb{E}[\bar{L}]\Phi(d_2) =& S_0 \\ \Leftrightarrow & & (S_0 + P)\Phi(d_1) - \mathbb{E}[\bar{L}]\Phi(d_2) =& S_0 \\ \Leftrightarrow & & S_0(\Phi(d_1) - 1) + P\Phi(d_1) - \mathbb{E}[\bar{L}]\Phi(d_2) =& 0 \end{aligned}$$

It is observed that:

$$\lim_{S_0\to\infty} d_1 = \lim_{\sigma\to 0} d_1 = \infty \quad \text{and} \quad \lim_{S_0\to\infty} d_2 = \lim_{\sigma\to 0} d_2 = \infty,$$

leading to:

$$\lim_{S_0\to\infty} \Phi(d_1) = \lim_{\sigma\to 0} \Phi(d_1) = 1 \quad \text{and} \quad \lim_{S_0\to\infty} \Phi(d_2) = \lim_{\sigma\to 0} \Phi(d_2) = 1.$$

Consequently, the equation simplifies to:

$$\begin{aligned} & & S_0(\Phi(d_1) - 1) + P\Phi(d_1) - \mathbb{E}[\bar{L}]\Phi(d_2) =& 0 \\ \Leftrightarrow & & P - \mathbb{E}[\bar{L}] =& 0 \\ \Leftrightarrow & & P =& \mathbb{E}[\bar{L}] \end{aligned}$$

□

Lemma 2

Proof. In the scenario where $\sigma \to 0$, uncertainty diminishes, eliminating jump risks. Consequently, the focus lies solely on the case where $S_0 \to \infty$. Without loss of generality, k, m and n can be fixed:

$$\exp\big(-(\tilde{\lambda}_1+\tilde{\lambda}_2+\tilde{\lambda})\big)\frac{(\tilde{\lambda}_1)^k}{k!}\frac{(\tilde{\lambda}_2)^m}{m!}\frac{(\tilde{\lambda})^n}{n!}$$
$$=$$
$$\exp\big(-\tilde{\lambda}_1\big)\frac{(\tilde{\lambda}_1)^k}{k!}\exp\big(-\tilde{\lambda}_2\big)\frac{(\tilde{\lambda}_2)^m}{m!}\exp\big(-\tilde{\lambda}\big)\frac{(\tilde{\lambda})^n}{n!}$$
$$=$$
$$\mathbb{P}_{\tilde{\lambda}_1}(k)\mathbb{P}_{\tilde{\lambda}_2}(m)\mathbb{P}_{\tilde{\lambda}}(n),$$

given the Poisson distribution of the jump occurrences. From the previous proof it is known that for $S_0 \to \infty$:

$$\Phi(d_1, t, k, m, n) = \Phi(d_2, t, k, m, n) = 1.$$

Thus, the option formula can be expressed as:

$$\begin{aligned}
C(Y_1(P), \bar{L}) = &\sum_k \sum_m \sum_n \mathbb{P}_{\tilde{\lambda}_1}(k)\mathbb{P}_{\tilde{\lambda}_2}(m)\mathbb{P}_{\tilde{\lambda}}(n) \\
&\times \Bigg[Y_0 \exp\Bigg(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1) + k\tilde{\alpha}_{11} + \frac{k\delta_{11}^2}{2} + n\tilde{\alpha}_1 + \frac{n\delta_1^2}{2}\Bigg) \\
&- \mathbb{E}[\bar{L}] \exp\Bigg(-(\tilde{\lambda}_2\tilde{\kappa}_{Z_2} + \tilde{\lambda}\tilde{\kappa}_2) + m\tilde{\alpha}_{22} + \frac{m\delta_{22}^2}{2} + n\tilde{\alpha}_2 + \frac{n\delta_2^2}{2}\Bigg) \Bigg]
\end{aligned}$$

Moreover:

$$\begin{aligned}
\exp(k\tilde{\alpha}_{11} + \frac{k\delta_{11}^2}{2}) &= \mathbb{E}[\exp(kZ_1] \\
\exp(m\tilde{\alpha}_{22} + \frac{m\delta_{22}^2}{2}) &= \mathbb{E}[\exp(mZ_2)] \\
\exp(n\tilde{\alpha}_1 + \frac{n\delta_1^2}{2}) &= \mathbb{E}[\exp(nY_1)] \\
\exp(n\tilde{\alpha}_2 + \frac{n\delta_2^2}{2}) &= \mathbb{E}[\exp(nY_2)],
\end{aligned}$$

and defining $\sum_{k,m,n} \mathbb{P}_{\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}}(k,m,n) = \sum_k \sum_m \sum_n \mathbb{P}_{\tilde{\lambda}_1}(k)\mathbb{P}_{\tilde{\lambda}_2}(m)\mathbb{P}_{\tilde{\lambda}}(n)$:

$$\begin{aligned}
C(Y_1(P),\bar{L}) =& \sum_{k,m,n} \mathbb{P}_{\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}}(k,m,n) \Bigg[Y_0 \exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big) \mathbb{E}[\exp(kZ_1)]\mathbb{E}[\exp(nY_1)] \\
& - \mathbb{E}[\bar{L}] \exp\Big(-(\tilde{\lambda}_2\tilde{\kappa}_{Z_2} + \tilde{\lambda}\tilde{\kappa}_2)\Big) \mathbb{E}[\exp(mZ_2)]\mathbb{E}[\exp(nY_2)] \Bigg] \\
=& \Bigg[Y_0 \exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big) \sum_{k,m,n} \mathbb{P}_{\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}}(k,m,n)\mathbb{E}[\exp(kZ_1)]\mathbb{E}[\exp(nY_1)] \\
& - \mathbb{E}[\bar{L}] \exp\Big(-(\tilde{\lambda}_2\tilde{\kappa}_{Z_2} + \tilde{\lambda}\tilde{\kappa}_2)\Big) \sum_{k,m,n} \mathbb{P}_{\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}}(k,m,n)\mathbb{E}[\exp(mZ_2)]\mathbb{E}[\exp(nY_2)] \Bigg]
\end{aligned}$$

The call option must equate to the initial equity, therefore:

$$\begin{aligned}
C(Y_1(P),\bar{L}) =& (S_0 + P) \exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big) \sum_{k,m,n} \mathbb{P}_{\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}}(k,m,n)\mathbb{E}[\exp(kZ_1)]\mathbb{E}[\exp(nY_1)] \\
& - \mathbb{E}[\bar{L}] \exp\Big(-(\tilde{\lambda}_2\tilde{\kappa}_{Z_2} + \tilde{\lambda}\tilde{\kappa}_2)\Big) \sum_{k,m,n} \mathbb{P}_{\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}}(k,m,n)\mathbb{E}[\exp(mZ_2)]\mathbb{E}[\exp(nY_2)] \\
=& S_0.
\end{aligned}$$

For the sake of a simpler overview, let's define:

$$J_1 = \sum_{k,m,n} \mathbb{P}_{\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}}(k,m,n)\mathbb{E}[\exp(kZ_1)]\mathbb{E}[\exp(nY_1)]$$

and:

$$J_2 = \sum_{k,m,n} \mathbb{P}_{\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}}(k,m,n)\mathbb{E}[\exp(mZ_2)]\mathbb{E}[\exp(nY_2)]$$

as a placeholder. Isolating the premium yields to:

$$\begin{aligned}
P =& \mathbb{E}[\bar{L}] \frac{\exp\Big(-(\tilde{\lambda}_2\tilde{\kappa}_{Z_2} + \tilde{\lambda}\tilde{\kappa}_2)\Big) J_2}{\exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big) J_1} + S_0 \frac{1 - \exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big) J_1}{\exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big) J_1} \\
=& \mathbb{E}[\bar{L}] \exp\Big(-(\tilde{\lambda}_2\tilde{\kappa}_{Z_2} - \tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}(\tilde{\kappa}_2 - \tilde{\kappa}_1))\Big) \frac{J_2}{J_1} + S_0 \frac{1 - \exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big) J_1}{\exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big) J_1}
\end{aligned}$$

Upon closer examination of J_1, its expression can be rephrased. Without loss of generality, the same restructuring applies to J_2 by substituting k and m:

$$\begin{aligned}
J_1 &= \sum_{k,m,n} \mathbb{P}_{\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}}(k,m,n)\mathbb{E}[\exp(kZ_1)]\mathbb{E}[\exp(nY_1)] \\
&= \underbrace{\sum_m \mathbb{P}_{\tilde{\lambda}_2}(m)}_{=1} \sum_k \mathbb{P}_{\tilde{\lambda}_1}(k)\mathbb{E}[\exp(kZ_1)] \sum_n \mathbb{P}_{\tilde{\lambda}}(n)\mathbb{E}[\exp(nY_1)].
\end{aligned}$$

Without loss of generality, the focus remains on $\sum_k \mathbb{P}_{\tilde{\lambda}_1}(k)\mathbb{E}[\exp(kZ_1)]$ with this equivalence extending to other components sharing a similar structure:

$$\begin{aligned}
\sum_k \mathbb{P}_{\tilde{\lambda}_1}(k)\mathbb{E}[\exp(kZ_1)] &= \sum_k \mathbb{P}_{\tilde{\lambda}_1}(k)\exp(k\tilde{\alpha}_{11} + k\frac{\delta_{22}^2}{2}) \\
&= \sum_k \mathbb{P}_{\tilde{\lambda}_1}(k)\exp(\tilde{\alpha}_{11} + \frac{\delta_{22}^2}{2})^k \\
&= \sum_k \mathbb{P}_{\tilde{\lambda}_1}(k)\mathbb{E}[\exp(Z_1)]^k \\
&= \sum_k \mathbb{P}_{\tilde{\lambda}_1}(k)\exp\big(k\ln(\mathbb{E}[\exp(Z_1)])\big)
\end{aligned}$$

Reflecting on the fact that the moment-generating function of a Poisson-distributed random variable x is defined as $M_X(u) = \mathbb{E}[\exp(uX)] = \sum_n \mathbb{P}(X = n)\exp(un)$, this results in:

$$\begin{aligned}
\sum_k \mathbb{P}_{\tilde{\lambda}_1}(k)\exp\big(k\ln(\mathbb{E}[\exp(Z_1)])\big) &= M_{N_1}\big(ln(M_{Z_1})\big) \\
&= \exp(\tilde{\lambda}_1(\exp\big(\ln(\mathbb{E}[\exp(Z_1)])\big) - 1)) \\
&= \exp(\tilde{\lambda}_1\underbrace{\big(\mathbb{E}[\exp(Z_1)] - 1\big)}_{\tilde{\kappa}_{Z1}}) \\
&= \exp(\tilde{\lambda}_1\tilde{\kappa}_{Z1}).
\end{aligned}$$

Summarized, it holds:

$$\begin{aligned}
J_1 &= \exp(\tilde{\lambda}_1\tilde{\kappa}_{Z1})\exp(\tilde{\lambda}\tilde{\kappa}_1) \\
J_2 &= \exp(\tilde{\lambda}_2\tilde{\kappa}_{Z2})\exp(\tilde{\lambda}\tilde{\kappa}_2)
\end{aligned}$$

Therefore, the following applies to the premium:

$$\begin{aligned}
P =& \mathbb{E}[\bar{L}]\exp\Big(-(\tilde{\lambda}_2\tilde{\kappa}_{Z_2} - \tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}(\tilde{\kappa}_2 - \tilde{\kappa}_1))\Big)\frac{\exp(\tilde{\lambda}_2\tilde{\kappa}_{Z2})\exp(\tilde{\lambda}\tilde{\kappa}_2)}{\exp(\tilde{\lambda}_1\tilde{\kappa}_{Z1})\exp(\tilde{\lambda}\tilde{\kappa}_1)} \\
&+ S_0\frac{1 - \exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big)\exp(\tilde{\lambda}_1\tilde{\kappa}_{Z1})\exp(\tilde{\lambda}\tilde{\kappa}_1)}{\exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big)\exp(\tilde{\lambda}_1\tilde{\kappa}_{Z1})\exp(\tilde{\lambda}\tilde{\kappa}_1)} \\
=& \mathbb{E}[\bar{L}]\exp\Big(-(\tilde{\lambda}_2\tilde{\kappa}_{Z_2} - \tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}(\tilde{\kappa}_2 - \tilde{\kappa}_1))\Big)\exp\Big((\tilde{\lambda}_2\tilde{\kappa}_{Z_2} - \tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}(\tilde{\kappa}_2 - \tilde{\kappa}_1))\Big) \\
&+ S_0\frac{1 - \exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big)\exp(\tilde{\lambda}_1\tilde{\kappa}_{Z1} + \tilde{\lambda}\tilde{\kappa}_1)}{\exp\Big(-(\tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}\tilde{\kappa}_1)\Big)\exp(\tilde{\lambda}_1\tilde{\kappa}_{Z1} + \tilde{\lambda}\tilde{\kappa}_1)} \\
=& \mathbb{E}[\bar{L}]
\end{aligned}$$

□