PERPETUAL AMERICAN OPTIONS CANCELLED AT LAST PASSAGE

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1. Introduction

Our primary goal is to derive the closed-form formula for the price of a perpetual American put option, terminated at the last-passage time of the underlying asset above a fixed level h. Formally, we seek the value function:

$$\overline{V}(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau}(K - S_\tau)^+ \mathbb{I}(\tau < \theta) | S_0 = s], \tag{1}$$

where $\mathcal{T}$ is a family of stopping times, S_t is the underlying asset price, $K > 0$ is the strike price, and r is the risk-free interest rate. In (1), the time-cap θ is the last-passage time of the asset price above fixed threshold $h > K$, that is,

$$\theta = \sup\{t \geq 0 \mid S_t \geq h\}. \tag{2}$$

In our model, the asset price follows a geometric spectrally negative Lévy process:

$$S_t = e^{X_t}, \tag{3}$$

where X_t is a spectrally negative Lévy process with $S_0 = s = e^x$. Hence X_t has stationary and independent increments without positive jumps. In particular, X_t can be, under a rik-neutral measure, a linear Brownian motion and in that case we have

$$X_t = \sigma B_t + \left(r - \frac{\sigma^2}{2}\right) t, \tag{4}$$

where r is a risk-free interest rate and $\sigma > 0$ is a volatility of the asset price. Then S_t is determined by the seminal Black-Scholes market. Another example of Lévy market, also analysed in this work, is the case of a jump-diffusion model of the asset price S_t. Then

$$X_t = x + \mu t + \sigma B_t - \sum_{k=1}^{N_t} U_k, \tag{5}$$

where $x = X_0 = \log(s)$, $\sigma > 0$, B_t is a Brownian motion, μ is a drift term, N_t is a homogeneous Poisson process with intensity λ, and U_k are i.i.d. exponential random variables with mean ρ^{-1}. We assume no dividends and take expectations under the martingale measure $\mathbb{P}$, ensuring $e^{X_t - rt}$ is a local martingale. Lévy markets offer a realistic representation of price dynamics, capturing skewness, asymmetry, and better model calibration (see, e.g., [Cont (2001), Cont and Tankov (2004)]).
To find the price (1) of our tiome-capped American option, we employ the 'guess and verify' method, hypothesizing the stopping time as the first downward crossing of a threshold $a > h$, calculating the value function using Lévy process fluctuation theory, and maximizing it with respect to a. We confirm our hypothesis by verifying the Hamilton-Jacobi-Bellman (HJB) equation. Then, we provide some numerical results.

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This paper includes results presented in [Palmowski and Stępniak (2023)]. Additionally, we briefly explore potential modifications of the model, which are better described in the papers [Palmowski and Stępniak (2024a)] and [Palmowski and Stępniak (2024b)].
The pricing of American options has been extensively studied; see [Detemple (2006)] for a review. This paper introduces a novel cancellation feature to the basic American contract, potentially valuable during volatile markets. Similar research includes [Gapeev at al. (2020)], which uses HJB equations for geometric Brownian motion. Our approach, suitable for nonlocal generators, offers efficiency through 'guess and verify'. We also relate our results to HJB equations.
Other relevant works reduce optimal stopping problems to free boundary problems. [Wu and Li (2022)] values American put options similarly but focuses on finite-time maturity within the Black-Scholes market. [Gapeev and Motairi (2022)] considers perpetual American cancelable options in the Black-Merton-Scholes market with stochastic cancellation times. [Gapeev and Li (2022a)], [Gapeev and Li (2022b)], and others study perpetual American options terminated at the last stock price extremities.

2. MAIN RESULT

To present the main result of this paper, we introduce required notations. For $r \geq 0$, we define the so-called scale function:

$$W^{(r)}(x) = \sum_{i=1}^{3} C_i e^{\eta_i x}, \tag{6}$$

where:

$$\eta_1 = 1, \qquad \eta_{2/3} = \frac{-1}{2(\rho\sigma^2 + \sigma^2)} \left(2\lambda + 2r + \rho^2\sigma^2 + \rho\sigma^2 + 2r\rho \pm 2\sqrt{\omega}\right) \tag{7}$$

and

$$\omega = \lambda^2 + \lambda(\rho+1)(2r+\sigma^2) + (\rho+1)^2 \left(r - \frac{1}{2}\rho\sigma^2\right)^2. \tag{8}$$

Furthermore,

$$C_1 = \frac{2(\eta_1 + \rho)}{\sigma^2(\eta_1 - \eta_2)(\eta_1 - \eta_3)}, \quad C_2 = \frac{2(\eta_2 + \rho)}{\sigma^2(\eta_2 - \eta_1)(\eta_2 - \eta_3)}, \tag{9}$$

$$C_3 = \frac{2(\eta_3 + \rho)}{\sigma^2(\eta_3 - \eta_1)(\eta_3 - \eta_2)}. \tag{10}$$

We show that the optimal exercise time is of the form

$$\tau_a = \inf\{t \geq 0 : S_t \leq a\} = \inf\{t \geq 0 : X_t \leq \log(a)\}. \tag{11}$$

The threshold a needs to be lower than the strike price K (and hence of the canceling threshold h), so that exercising the option can be profitable to the holder. Therefore,

$$0 < a < K. \tag{12}$$

We denote

$$G(s) = (K - s)^+ \left(\left(\frac{h}{s}\right)^\alpha \wedge 1\right), \tag{13}$$

where

$$\alpha = \frac{\rho}{2} + \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\rho}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\lambda}{\sigma^2}}. \tag{14}$$

The main result of [Palmowski and Stępniak (2023)] is as follows.

Theorem 1. *The price of the perpetual American cancelable put option defined in* (1) *equals*

$$\overline{V}(s) = \frac{\sigma^2}{2}\left[W^{(r)'}\left(\log\frac{s}{a^*}\right) - \Phi(r)W^{(r)}\left(\log\frac{s}{a^*}\right)\right]G(a^*)$$

$$+\left[Z^{(r)}\left(\log\frac{s}{a^*}\right) - \frac{\sigma^2}{2}W^{(r)'}\left(\log\frac{s}{a^*}\right) - W^{(r)}\left(\log\frac{s}{a^*}\right)\left(\frac{r}{\Phi(r)} - \frac{\Phi(r)\sigma^2}{2}\right)\right]$$
$$\times\int_0^\infty \rho e^{-\rho y} G\left(a^* e^{-y}\right)\mathrm{d}y$$

and τ_{a^*} *defined in* (11) *is the optimal stopping rule for the optimal stopping threshold*

$$a^* = \frac{K\left(\frac{\sigma^2}{2}\sum_{i=2}^3 C_i\eta_i(\eta_i-1) + \alpha + \frac{\rho}{\rho-\alpha}\sum_{i=2}^3 C_i\eta_i\left[r\left(\frac{1}{\eta_i}-1\right) - \frac{\sigma^2}{2}(\eta_i-1)\right]\right)}{\frac{\sigma^2}{2}\sum_{i=2}^3 C_i\eta_i(\eta_i-1) - (1-\alpha) + \frac{\rho}{\rho-\alpha+1}\sum_{i=2}^3 C_i\eta_i\left[r\left(\frac{1}{\eta_i}-1\right) - \frac{\sigma^2}{2}(\eta_i-1)\right]}. \tag{15}$$

3. NUMERICAL ANALYSIS

In this part we perform a numerical analysis based on Theorem 1. We choose a set of parameters: $h = 120$, $s = S_0 = 100$, $K = 100$, $r = 10\%$, $\sigma^2 = 0.25$, $\lambda = 5$ and $\rho = 4$. We start by finding the optimal threshold a^*. With formula (15), we obtain $a^* \approx 54.78$. Then, we use Theorem 1 to find the fair price of the cancelable option, which yields $\overline{V}(s) \approx 38.41$. In Figure 1, we illustrate the behavior of the payoff and price functions of the cancelable put option, depending on the initial underlying asset price. The smooth fit of these functions is clearly visible for $S = a^*$.

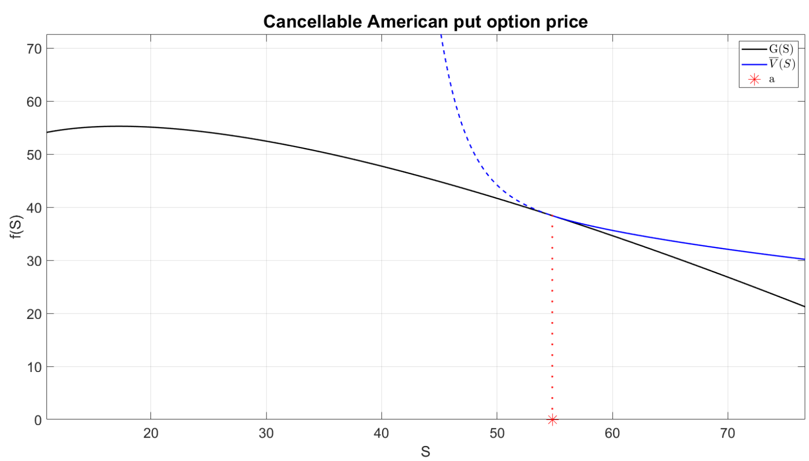

FIGURE 1. Smooth fit of the payoff and the price functions for: $\rho = 4$, $\sigma^2 = 0.25$, $r = 0.1$, $\lambda = 2$, $K = 100$, $h = 120$, $S_0 = 100$. $f(S)$ denotes the value function f where $f(S) = G(S)$ or $f(S) = \overline{V}(S)$.

4. FURTHER WORK

Future research will extend the current model to incorporate additional constraints on the time to maturity. Specifically, we propose a modified Least Squares Monte Carlo (LSMC) method suitable for this type of contract within general spectrally negative Lévy markets. We will detail the methodology, establish proof of convergence, and present numerical results. These findings will be elaborated in the forthcoming work, [Palmowski and Stępniak (2024a)], currently under review. Additionally, in [Palmowski and Stępniak (2024b)], we will investigate options terminated by the first passage of a drawdown process. In this context, the option is capped when the underlying asset price drops by a predetermined threshold below its historical maximum. Utilizing the 'guess and verify' technique, we will identify the optimal stopping time and derive a closed-form formula for the option price. In sum, our research opens several avenues for extending the analysis of American options with complex cancellation features, offering both theoretical insights and practical tools for financial practitioners.

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